ON ENDOMORPHISM ALGEBRAS OF SEPARABLE MONOIDAL FUNCTORS

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ABSTRACT. We show that the (co)endomorphism algebra of a sufficiently separable "fibre" functor into $\mathbf{Vect}_k$, for k a field of characteristic 0, has the structure of what we call a "unital" von Neumann core in $\mathbf{Vect}_k$. For $\mathbf{Vect}_k$, this particular notion of algebra is weaker than that of a Hopf algebra, although the corresponding concept in **Set** is again that of a group.

1. INTRODUCTION

Let $(\mathscr{C}, \otimes, I, c)$ be a symmetric (or just braided) monoidal category. Recall that an *algebra* in $\mathscr{C}$ is an object $A \in \mathscr{C}$ equipped with a multiplication $\mu : A \otimes A \longrightarrow A$ and a unit $\eta : I \longrightarrow A$ satisfying $\mu_3 = \mu(1 \otimes \mu) = \mu(\mu \otimes 1) : A^{\otimes 3} \longrightarrow A$ (associativity) and $\mu(\eta \otimes 1) = 1 = \mu(1 \otimes \eta) : A \longrightarrow A$ (unit conditions). Dually, a *coalgebra* in $\mathscr{C}$ is an object $C \in \mathscr{C}$ equipped with a comultiplication $\delta : C \longrightarrow C \otimes C$ and a counit $\epsilon : C \longrightarrow I$ satisfying $\delta_3 = (1 \otimes \delta)\delta = (\delta \otimes 1)\delta : C \longrightarrow C^{\otimes 3}$ (coassociativity) and $(\epsilon \otimes 1)\delta = 1 = (1 \otimes \epsilon)\delta : C \longrightarrow C$ (counit conditions).

A *very weak bialgebra* in $\mathscr{C}$ is an object $A \in \mathscr{C}$ with both the structure of an algebra and a coalgebra in $\mathscr{C}$ related by the axiom

$$\delta\mu = (\mu \otimes \mu)(1 \otimes c \otimes 1)(\delta \otimes \delta) : A \otimes A \longrightarrow A \otimes A.$$

For example, any k-bialgebra or weak k-bialgebra is a very weak bialgebra in this sense (for $\mathscr{C} = \mathbf{Vect}_k$). The structure A is then called a *von Neumann core* in $\mathscr{C}$ if it also has an antipode $S : A \longrightarrow A$ satisfying the axiom

$$\mu_3(1 \otimes S \otimes 1)\delta_3 = 1 : A \longrightarrow A.$$

For example, the set of all finite paths of edges in a (row-finite) graph algebra [8] forms a von Neumann core in $\mathscr{C} = \mathbf{Set}$, and so does any group in **Set**.

Since groups A in **Set** are characterized by the (stronger) axiom

$$(\dagger) \qquad 1 \otimes \eta = (1 \otimes \mu)(1 \otimes S \otimes 1)\delta_3 : A \longrightarrow A \otimes A,$$

a very weak bialgebra A satisfying (†), in the general $\mathscr{C}$, will be called a *unital* von Neumann core in $\mathscr{C}$. Such a unital core A always has a left inverse, namely $(1 \otimes \mu)(1 \otimes S \otimes 1)(\delta \otimes 1)$, to the "fusion" operator

$$(1 \otimes \mu)(\delta \otimes 1) : A \otimes A \longrightarrow A \otimes A,$$

and the latter satisfies the fusion equation [9]. Any Hopf algebra in $\mathscr{C}$ satisfies the axiom (†), and in this article we are mainly interested in producing a unital von

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Neumann core, namely $\mathrm{End}^{\vee}U$, associated to a certain type of monoidal functor U into $\mathbf{Vect}_k$. However, it will not be the case that all unital von Neumann cores in $\mathbf{Vect}_k$ can be reproduced as such.

We will tacitly assume throughout the article that the ground category [7] is $\mathbf{Vect} = \mathbf{Vect}_k$, for k a field of characteristic 0, so that the categories and functors considered here are all k-linear (although any reasonable category $[\mathscr{D}, \mathbf{Vect}]$ of parameterized vector spaces would suffice). We denote by $\mathbf{Vect}_f$ the full subcategory of $\mathbf{Vect}$ consisting of the finite dimensional vector spaces, and we further suppose that $(\mathscr{C}, \otimes, I, c)$ is a braided monoidal category with a "fibre" functor

$$U : \mathscr{C} \longrightarrow \mathbf{Vect}$$

which has both a monoidal structure (U, r, r_0) and a comonoidal structure (U, i, i_0). We call U *separable*[1] if $ri = 1$ and $i_0 r_0 = \dim(UI) \cdot 1$; i.e., for all $A, B \in \mathscr{C}$, the diagrams

$$\begin{tikzcd} U(A \otimes B) \arrow[r, "i"] \arrow[dr, "1"'] & UA \otimes UB \arrow[d, "r"] \\ & U(A \otimes B) \end{tikzcd} \qquad \begin{tikzcd} I \arrow[r, "r_0"] \arrow[dr, "\dim UI \cdot 1"'] & UI \arrow[d, "i_0"] \\ & I \end{tikzcd}$$

commute.

First we produce an algebra structure (μ, η) on

$$\mathrm{End}^{\vee}U = \int^{C} UC^{*} \otimes UC$$

using the monoidal and comonoidal structures on U. Secondly, we suppose that $\mathscr{C}$ has a suitable small generating set $\mathscr{A}$ of objects, and produce a coalgebra structure (δ, ϵ) on $\mathrm{End}^{\vee}U$ when each value UA, $A \in \mathscr{A}$, is finite dimensional. Finally, we assume that U is equipped with a natural non-degenerate form

$$U(A^{*}) \otimes UA \longrightarrow k$$

suitably related to the evaluation and coevaluation maps of $\mathscr{C}$ and $\mathbf{Vect}_f$, where each $A \in \mathscr{A}$ has a $\otimes$-dual A^{*} in $\mathscr{C}$ which again lies in $\mathscr{A}$. This last assumption is sufficient to provide $\mathrm{End}^{\vee}U$ with an antipode so that it becomes a unital von Neumann core in the above sense.

By way of examples, we note that many separable monoidal functors are constructable from separable monoidal categories; i.e., from monoidal categories $\mathscr{C}$ for which the tensor product map

$$\otimes : \mathscr{C}(A, B) \otimes \mathscr{C}(C, D) \longrightarrow \mathscr{C}(A \otimes C, B \otimes D)$$

is a naturally split epimorphism (as is the case for some finite cartesian products such as $\mathbf{Vect}_f^n$). A closely related source of examples is the notion of a weak dimension functor on $\mathscr{C}$ (cf. [5]); this is a comonoidal functor

$$(d, i, i_0) : \mathscr{C} \longrightarrow \mathbf{Set}_f$$

[1] Strictly, we should also require the conditions

$$(r \otimes 1)(1 \otimes i) = ir : UA \otimes U(B \otimes C) \longrightarrow U(A \otimes B) \otimes UC, \text{ and}$$
$$(1 \otimes r)(i \otimes 1) = ir : U(A \otimes B) \otimes UC \longrightarrow UA \otimes U(B \otimes C)$$

in order for U to be called "separable", but we do not need these here.

for which the comonoidal transformation components

$$i = i_{C,D} : d(C \otimes D) \longrightarrow dC \times dD$$

are injective functions, while the unique map $i_0 : dI \longrightarrow 1$ is surjective. Various examples are described at the conclusion of the paper.

We suppose the reader is familiar to some extent with the standard references on the problem when restricted to the case of U strong monoidal.

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2. The algebraic structure on $\mathrm{End}^\vee U$

If $\mathscr{C}$ is a (k-linear) monoidal category and

$$U : \mathscr{C} \longrightarrow \mathbf{Vect}$$

has a monoidal structure (U, r, r_0) and a comonoidal structure (U, i, i_0), then $\mathrm{End}^\vee U$ has an associative and unital k-algebra structure whose multiplication μ is the composite map

$$\begin{tikzcd}
\int^C UC^* \otimes UC \otimes \int^D UD^* \otimes UD \arrow[rr, "\mu"] \arrow[d, "\cong"'] & & \int^B UB^* \otimes UB \\
\int^{C,D} UC^* \otimes UD^* \otimes UC \otimes UD \arrow[d, "\mathrm{can}"'] & & \\
\int^{C,D} (UC \otimes UD)^* \otimes UC \otimes UD \arrow[rr, "\int i^* \otimes r"'] & & \int^{C,D} U(C \otimes D)^* \otimes U(C \otimes D) \arrow[uu, "\int^{\otimes}"']
\end{tikzcd}$$

while the unit η is given by

$$\begin{tikzcd}
k \arrow[r, "\eta"] \arrow[d, "\cong"'] & \int^C UC^* \otimes UC \\
k^* \otimes k \arrow[r, "i_0^* \otimes r_0"'] & UI^* \otimes UI. \arrow[u, "\mathrm{copr}_{C=I}"']
\end{tikzcd}$$

The associativity and unit axioms for $(\mathrm{End}^\vee U, \mu, \eta)$ now follow directly from the corresponding associativity and unit axioms for (U, r, r_0) and (U, i, i_0). An augmentation ϵ is given by

$$\begin{tikzcd}
\int^C UC^* \otimes UC \arrow[r, "\epsilon"] & k \\
UD^* \otimes UD \arrow[u, "\mathrm{copr}_{C=D}"] \arrow[ur, "\mathrm{ev}"'] &
\end{tikzcd}$$

in $\mathbf{Vect}$, where $\epsilon\eta = \dim UI \cdot 1$.

We also observe that the coend

$$\mathrm{End}^\vee U = \int^C UC^* \otimes UC$$

actually exists in **Vect** if $\mathscr{C}$ contains a small full subcategory $\mathscr{A}$ with the property that the family

$$\{Uf : UA \longrightarrow UC \mid f \in \mathscr{C}(A, C), A \in \mathscr{A}\}$$

is epimorphic in **Vect** for each object $C \in \mathscr{C}$. In fact, we shall use the stronger condition that the maps

$$\alpha_C : \int^{A \in \mathscr{A}} \mathscr{C}(A, C) \otimes UA \longrightarrow UC$$

should be isomorphisms, not just epimorphisms. This stronger condition implies that we can effectively replace $\int^{C \in \mathscr{C}}$ by $\int^{A \in \mathscr{A}}$ since

$$\begin{aligned} \int^{C} UC^* \otimes UC &\cong \int^{C} UC^* \otimes (\int^{A} \mathscr{C}(A, C) \otimes UA) \\ &\cong \int^{A} UA^* \otimes UA \end{aligned}$$

by the Yoneda lemma.

If we furthermore ask that each value UA be finite dimensional for A in $\mathscr{A}$, then

$$\mathrm{End}^{\vee} U \cong \int^{A \in \mathscr{A}} UA^* \otimes UA$$

is canonically a k-coalgebra with counit the augmentation ϵ, and comultiplication δ given by

$$\begin{tikzcd} \int^{A} UA^* \otimes UA \arrow[r, "\delta"] & \int^{A} UA^* \otimes UA \otimes \int^{A} UA^* \otimes UA \\ UA^* \otimes UA \arrow[u, "\mathrm{copr}"] \arrow[r, "1 \otimes n \otimes 1"'] & UA^* \otimes UA \otimes UA^* \otimes UA, \arrow[u, "\mathrm{copr} \otimes \mathrm{copr}"'] \end{tikzcd}$$

where n denotes coevaluation in $\mathbf{Vect}_f$.

Proposition 2.1. *If* U *is separable then* $\mathrm{End}^{\vee} U$ *satisfies the* k*-bialgebra axiom*

$$\begin{tikzcd} \mathrm{End}^{\vee} U \otimes \mathrm{End}^{\vee} U \arrow[r, "\delta \otimes \delta"] \arrow[dd, "\mu"] & (\mathrm{End}^{\vee} U)^{\otimes 4} \arrow[d, "1 \otimes c \otimes 1"] \\ & (\mathrm{End}^{\vee} U)^{\otimes 4} \arrow[d, "\mu \otimes \mu"] \\ \mathrm{End}^{\vee} U \arrow[r, "\delta"] & \mathrm{End}^{\vee} U \otimes \mathrm{End}^{\vee} U. \end{tikzcd}$$

Proof. Let $\mathscr{B}$ denote the monoidal full subcategory of $\mathscr{C}$ generated by $\mathscr{A}$ (we will essentially replace $\mathscr{C}$ by this small category $\mathscr{B}$). Then, for all C, D in $\mathscr{B}$, we have, by induction on the tensor lengths of C and D, that $U(C \otimes D)$ is finite dimensional since it is a retract of $UC \otimes UD$. Moreover, we have

$$\int^{A \in \mathscr{A}} UA^* \otimes UA \;\cong\; \int^{B \in \mathscr{B}} UB^* \otimes UB$$

by the Yoneda lemma, since the natural family

$$\alpha_B : \int^{A \in \mathscr{A}} \mathscr{C}(A,B) \otimes UA \longrightarrow UB$$

is an isomorphism for all $B \in \mathscr{B}$. Since $ri = 1$, the triangle

$$\begin{tikzcd}
 & (UC \otimes UD) \otimes (UC \otimes UD)^* \arrow[d, "r \otimes i^*"] \\
k \arrow[ur, "n"] \arrow[r, "n"'] & U(C \otimes D) \otimes U(C \otimes D)^*
\end{tikzcd}$$

commutes in $\mathbf{Vect}_f$, where n denotes the coevaluation maps. The asserted bialgebra axiom then holds on $\mathrm{End}^{\vee}U$ since it reduces to the following diagram on filling in the definitions of μ and δ (where, for the moment, we have dropped the symbol "$\otimes$"):

$$\begin{tikzcd}
UC\,UC^*\,UD\,UD^* \arrow[r, "1\ n\ 1\ 1\ n\ 1"] \arrow[d, "\cong"'] & UC\,(UC\,UC^*)\,UC^*\,UD\,(UD\,UD^*)\,UD^* \arrow[d, "\cong"] \\
UC\,UD\,UC^*\,UD^* \arrow[d, "\cong"'] & UC\,UD\,UC\,UD\,UC^*\,UD^*\,UC^*\,UD^* \arrow[d, "\cong"] \\
UC\,UD\,(UC\,UD)^* \arrow[r, "1\ n\ 1"] \arrow[d, "r\ i^*"'] & UC\,UD\,UC\,UD\,(UC\,UD)^*\,(UC\,UD)^* \arrow[d, "r\ r\ i^*\ i^*"] \\
U(C\ D)\,U(C\ D)^* \arrow[r, "1\ n\ 1"] & U(C\ D)\,U(C\ D)\,U(C\ D)^*\,U(C\ D)^*
\end{tikzcd}$$

for all $C, D \in \mathscr{B}$. □

Notably the bialgebra axiom

$$\begin{tikzcd}
\mathrm{End}^{\vee}U \otimes \mathrm{End}^{\vee}U \arrow[rr, "\mu"] \arrow[dr, "\epsilon \otimes \epsilon"'] & & \mathrm{End}^{\vee}U \arrow[dl, "\epsilon"] \\
 & k &
\end{tikzcd}$$

does not hold in general, while the form of the axiom

$$\begin{tikzcd}
\mathrm{End}^{\vee}U \arrow[rr, "\delta"] & & \mathrm{End}^{\vee}U \otimes \mathrm{End}^{\vee}U \\
 & k \arrow[ul, "\eta"] \arrow[ur, "\eta \otimes \eta"'] &
\end{tikzcd}$$

holds where we multiply δ by $\dim UI$.

The k-bialgebra axiom established in the above proposition implies that the "fusion" operator $(1 \otimes \mu)(\delta \otimes 1) : A \otimes A \longrightarrow A \otimes A$ satisfies the fusion equation (see [9] for details).

The k-linear dual of $\mathrm{End}^\vee U$ is of course

$$[\int^C UC^* \otimes UC, k] \cong \int_C [UC^*, UC^*]$$

which is the endomorphism k-algebra of the functor

$$U(-)^* : \mathscr{C}^{\mathrm{op}} \longrightarrow \mathbf{Vect}.$$

If ob$\mathscr{A}$ is finite, so that

$$\int^A UA^* \otimes UA$$

is finite dimensional, then

$$\int_C [UC^*, UC^*] \cong \int_A [UA^*, UA^*]$$

is also a k-coalgebra.

3. The unital von Neumann antipode

We now take $(\mathscr{C}, \otimes, I, c)$ to be a braided monoidal category and $\mathscr{A} \subset \mathscr{C}$ to be a small full subcategory of $\mathscr{C}$ for which the monoidal and comonoidal functor $U : \mathscr{C} \longrightarrow \mathbf{Vect}$ induces

$$U : \mathscr{A} \longrightarrow \mathbf{Vect}_f$$

on restriction to $\mathscr{A}$. We suppose that $\mathscr{A}$ is such that

- the identity I of $\otimes$ lies in $\mathscr{A}$, and each object of $A \in \mathscr{A}$ has a $\otimes$-dual A^* lying in $\mathscr{A}$.

With respect to U, we suppose $\mathscr{A}$ has the properties

- "U-irreducibility": $\mathscr{A}(A, B) \neq 0$ implies $\dim UA = \dim UB$ for all $A, B \in \mathscr{A}$,
- "U-density": the canonical map
$$\alpha_C : \int^{A \in \mathscr{A}} \mathscr{C}(A, C) \otimes UA \longrightarrow UC$$
is an isomorphism for all $C \in \mathscr{C}$,
- "U-trace": each object of $\mathscr{A}$ has a U-trace in $\mathscr{C}(I, I)$, where by U-trace of $A \in \mathscr{A}$ we mean an isomorphism $d(A)$ in $\mathscr{C}(I, I)$ such that the following two diagrams commute.

$$\begin{tikzcd} I \arrow[r, "d(A)"] \arrow[d, "n"'] & I \\ A \otimes A^* \arrow[r, "c"] & A^* \otimes A \arrow[u, "e"'] \end{tikzcd} \qquad \begin{tikzcd} k \arrow[r, "\dim UA"] \arrow[d, "r_0"'] & k \arrow[d, "r_0"] \\ UI \arrow[r, "\dim UI \cdot U(d(A))"] & UI \end{tikzcd}$$

We shall assume $\dim UI \neq 0$ so that the latter assumption implies $\dim UA \neq 0$, for all $A \in \mathscr{A}$.

We require also a natural isomorphism

$$u = u_A : U(A^*) \xrightarrow{\cong} UA^*$$

such that

$$(n, r, r_0) \qquad \begin{array}{ccc} k & \xrightarrow{r_0} & UI \\ \downarrow n & & \downarrow Un \\ UA \otimes UA^* & & U(A \otimes A^*) \\ & \searrow^{1\otimes u^{-1}} \quad \nearrow_{r} & \\ & UA \otimes U(A^*) & \end{array}$$

commutes, and

$$(e, i, i_0) \qquad \begin{array}{ccc} UI & \xrightarrow{i_0} & k \\ \uparrow Ue & & \uparrow e \\ U(A^* \otimes A) & & UA^* \otimes UA \\ & \searrow^{i} \quad \nearrow_{u\otimes 1} & \\ & U(A^*) \otimes UA & \end{array}$$

commutes. This means that U "preserves duals" when restricted to $\mathscr{A}$.

An endomorphism

$$\sigma : \mathrm{End}^{\vee}U \longrightarrow \mathrm{End}^{\vee}U$$

may be defined by components

$$\begin{array}{ccc} \int^A UA^* \otimes UA & \xrightarrow{\sigma} & \int^A UA^* \otimes UA \\ \uparrow \mathrm{copr} & & \uparrow \mathrm{copr} \\ UA^* \otimes UA & \xrightarrow{\sigma_A} & U(A^*)^* \otimes U(A^*), \end{array}$$

each σ_A being given by commutativity of

$$\begin{array}{ccc} UA^* \otimes UA & \xrightarrow{\sigma_A} & U(A^*)^* \otimes U(A^*) \\ \downarrow 1\otimes\rho & & \uparrow c \\ UA^* \otimes UA^{**} & \xrightarrow{u^{-1}\otimes u^*} & U(A^*) \otimes U(A^*)^* \end{array}$$

where ρ denotes the canonical isomorphism from a finite dimensional vector space to its double dual. Clearly each component σ_A is invertible.

Theorem 3.1. *Let $\mathscr{C}$, $\mathscr{A}$, and U be as above, and suppose that U is braided and separable as a monoidal functor. Then there is an invertible antipode S on $\mathrm{End}^{\vee}U$ such that $(\mathrm{End}^{\vee}U, \mu, \eta, \delta, \epsilon, S)$ is a unital von Neumann core in $\mathbf{Vect}_k$.*

Proof. A family of maps $\{S_A \mid A \in \mathscr{A}\}$ is defined by

$$S_A = \dim UI \cdot (\dim UA)^{-1} \cdot \sigma_A.$$

Then, by the U-irreducibility assumption on the category $\mathscr{A}$, this family induces an invertible endomorphism S on the coend

$$\mathrm{End}^{\vee}U \cong \sum_{n=1}^{\infty} \int^{A\in\mathscr{A}_n} UA^* \otimes UA,$$

where $\mathscr{A}_n$ is the full subcategory of $\mathscr{A}$ determined by $\{A \mid \dim UA = n\}$. We now take S to be the antipode on $\mathrm{End}^{\vee}U$ and check that

$$1 \otimes \eta = (1 \otimes \mu)(1 \otimes S \otimes 1)\delta_3.$$

From the definition of μ and δ, we require commutativity of the exterior of the following diagram (where, again, we have dropped the symbol "$\otimes$"):

$$\begin{tikzcd}
UA^*\,UA\,UA^*\,UA\,UA^*\,UA \arrow[rr, "1\,1\,S_A\,1\,1"] & & UA^*\,UA\,U(A^*)^*\,U(A^*)\,UA^*\,UA \arrow[ddd, "\cong"] \\
& UA^*\,UA\,UA\,UA^*\,UA^*\,UA \arrow[ul, "1\,1\,c\,1\,1"'] & \\
& UA^*\,UA\,UA\,UA^*\,UA^*\,UA \arrow[u, "1\,1\,1\,c\,1"'] \arrow[l, phantom, "(1)"] \arrow[r, phantom, "(2)"] & \\
UA^*\,UA\,UA^*\,UA \arrow[uuu, "1\,n\,1\,1\,1"] \arrow[ur, "1\,1\,n\,1\,1"] \arrow[rr, "1\,1\,e^*\,1\,1"] & & UA^*\,UA\,UA^{**}\,UA^*\,UA^*\,UA \arrow[d, "\cong"] \\
& & UA^*\,UA\,U(A^*)^*\,U(A^*)\,UA^*\,UA \arrow[d, "\cong"] \\
UA^*\,UA \arrow[uu, "1\,n\,1"] \arrow[r, phantom, "(3)"] & & UA^*\,UA\,(U(A^*)\,UA)^*\,U(A^*)\,UA \arrow[d, "1\,1\,i^*\,r"] \\
& & UA^*\,UA\,U(A^*\,A)^*\,U(A^*\,A) \arrow[d, "1\,1\,\mathrm{copr}"] \\
UA^*\,UA\,I \arrow[uu, "\cong"] \arrow[rr, "1\,1\,\eta"] & & UA^*\,UA\,\int^B UB^*\,UB
\end{tikzcd}$$

The region labelled by (1) commutes on composition with $1 \otimes n \otimes 1$ since

$$\begin{tikzcd}
k \arrow[r, "n"] \arrow[ddd, "n"'] & UA \otimes UA^* \arrow[d, "1\otimes n\otimes 1"] \\
& UA \otimes UA \otimes UA^* \otimes UA^* \arrow[d, "1\otimes 1\otimes c"] \\
& UA \otimes UA \otimes UA^* \otimes UA^* \arrow[d, "1\otimes c\otimes 1"] \\
UA \otimes UA^* \arrow[r, "n\otimes 1\otimes 1"] & UA \otimes UA^* \otimes UA \otimes UA^*
\end{tikzcd}$$

commutes (choose a basis for UA). The region labelled by (2) now commutes by inspection of:

$$\begin{tikzcd}
UA\otimes UA^*\otimes UA\otimes UA^* \arrow[rr, "1\otimes\sigma_A\otimes 1"] \arrow[dddrr, "1\otimes 1\otimes\rho\otimes 1"] & & UA\otimes U(A^*)^*\otimes U(A^*)\otimes UA^* \\
& & UA\otimes U(A^*)\otimes U(A^*)^*\otimes UA^* \arrow[u, "1\otimes c\otimes 1"'] \\
UA\otimes UA\otimes UA^*\otimes UA^* \arrow[uu, "1\otimes c\otimes 1"] & & \\
& & UA\otimes UA^*\otimes UA^{**}\otimes UA^* \arrow[uu, "1\otimes u^{-1}\otimes u^*\otimes 1"'] \\
UA\otimes UA\otimes UA^*\otimes UA^* \arrow[uu, "1\otimes 1\otimes c"] \arrow[ddrr, "1\otimes\rho\otimes 1\otimes 1"] & & \\
& & UA\otimes UA^{**}\otimes UA^*\otimes UA^* \arrow[uu, "1\otimes c\otimes 1"'] \\
UA\otimes UA^* \arrow[uu, "1\otimes n\otimes 1"] \arrow[rr, "1\otimes e^*\otimes 1"] & & UA\otimes UA^{**}\otimes UA^*\otimes UA^* \arrow[u, "1\otimes 1\otimes c"']
\end{tikzcd}$$

where the top leg of (2) has been rescaled by a factor of $(\dim UI)^{-1}\cdot\dim UA$.

From the definition of the U-trace $d(A)$ of $A\in\mathscr{A}$, we have that

$$\begin{tikzcd}
k \arrow[r, "\dim UI\cdot(\dim UA)^{-1}"] \arrow[d, "r_0"'] & k \arrow[d, "r_0"] \\
UI \arrow[r, "U(d(A)^{-1})"'] & UI
\end{tikzcd}$$

commutes, so that the exterior of

$$\begin{tikzcd}
& UA\otimes UA^* \arrow[dr, "r"] & \\
UA\otimes UA^* \arrow[ur, "1\otimes u^{-1}"] \arrow[rr, phantom, "{(n,r,r_0)}"] & & U(A\otimes A^*) \\
k \arrow[u, "n"] \arrow[rr, "r_0"'] & & UI \arrow[u, "Un"'] \\
k \arrow[u, "\dim UI\cdot(\dim UA)^{-1}"] \arrow[rr, "r_0"] & & UI \arrow[u, "U(d(A)^{-1})"']
\end{tikzcd}$$

commutes.

Thus the region labelled by (3), with the top leg rescaled by the factor $\dim UI \cdot (\dim UA)^{-1}$, commutes on examination of the following diagram:

$$\begin{tikzcd}
& & (UA^* \otimes UA)^* \otimes UA^* \otimes UA \arrow[drr, "(u\otimes 1)^*\otimes(u^{-1}\otimes 1)"] & & \\
k^* \otimes UA^* \otimes UA \arrow[urr, "e^*\otimes 1\otimes 1"] \arrow[dr, "1\otimes u^{-1}\otimes 1"] \arrow[rrrr, phantom, "(e,i,i_0)"] & & & & (U(A^*) \otimes UA)^* \otimes U(A^*) \otimes UA \arrow[ddd, "i^*\otimes r"] \\
& k^* \otimes U(A^*) \otimes UA \arrow[dr, "i_0^*\otimes r"] & & & \\
& & UI^* \otimes U(A^* \otimes A) \arrow[drr, "Ue^*\otimes 1"] \arrow[dddr, "1\otimes Ue"] & & \\
k^* \otimes UA \otimes UA^* \arrow[uuu, "1\otimes c"] \arrow[dr, "1\otimes 1\otimes u^{-1}"] & & & & U(A^* \otimes A)^* \otimes U(A^* \otimes A) \arrow[ddd, "\mathrm{copr}"] \\
& k^* \otimes UA \otimes U(A^*) \arrow[uuu, "1\otimes c"] \arrow[dr, "i_0^*\otimes r"] \arrow[ur, phantom, "(*)"] & & & \\
& & UI^* \otimes U(A \otimes A^*) \arrow[uuu, "1\otimes Uc"] & UI^* \otimes UI \arrow[dr, "\mathrm{copr}"] & \\
k^* \otimes k \arrow[uuu, "1\otimes \dim UI\cdot(\dim UA)^{-1}\cdot n"] \arrow[rr, swap, "i_0^*\otimes r"] \arrow[ur, phantom, "(n,r,r_0)"] & & UI^* \otimes UI \arrow[u, "1\otimes U(d(A)^{-1})\cdot Un"] \arrow[ur, "1"] \arrow[rr, swap, "\mathrm{copr}"] & & \int^B UB^* \otimes UB
\end{tikzcd}$$

whose commutativity depends on the hypothesis that (U, r, r_0) is braided monoidal in order for

$$\begin{tikzcd}
UA \otimes U(A^*) \arrow[r, "c"] \arrow[d, swap, "r"] \arrow[dr, phantom, "(*)"] & U(A^*) \otimes UA \arrow[d, "r"] \\
U(A \otimes A^*) \arrow[r, swap, "Uc"] & U(A^* \otimes A)
\end{tikzcd}$$

to commute. $\square$

4. The fusion operator

Let $E = \mathrm{End}^\vee U$. The unital von Neumann axiom on E implies that the fusion operator

$$f = (1 \otimes \mu)(\delta \otimes 1) : E \otimes E \longrightarrow E \otimes E$$

has a left inverse, namely $g = (1 \otimes \mu)(1 \otimes S \otimes 1)(\delta \otimes 1)$. For this we consider the following diagram:

$$\begin{tikzcd}
E \otimes E \arrow[r, "\delta\otimes 1"] \arrow[dr, "\delta_3\otimes 1"'] \arrow[ddd, "1\otimes\eta\otimes 1"] \arrow[dddd, bend right, "1"'] & E^{\otimes 3} \arrow[r, "1\otimes\mu"] \arrow[d, shift right, "1\otimes\delta\otimes 1"'] \arrow[d, shift left, "\delta\otimes 1\otimes 1"] & E \otimes E \arrow[dd, "\delta\otimes 1"] \\
 & E^{\otimes 4} \arrow[dr, "1\otimes 1\otimes\mu"] \arrow[dd, "1\otimes S\otimes 1\otimes 1"] & \\
 & & E^{\otimes 3} \arrow[dd, "1\otimes S\otimes 1"] \\
E^{\otimes 3} \arrow[d, "1\otimes\mu"] & E^{\otimes 4} \arrow[l, "1\otimes\mu\otimes 1"'] \arrow[dr, "1\otimes 1\otimes\mu"] & \\
E \otimes E & & E^{\otimes 3}. \arrow[ll, "1\otimes\mu"']
\end{tikzcd}$$

In particular $f = (1 \otimes \mu)(\delta \otimes 1)$ is a partial isomorphism, i.e., $fgf = f$ and $gfg = g$.

5. Examples of separable monoidal functors in the present context

Unless otherwise indicated, categories, functors, and natural transformations shall be k-linear, for k a suitable field.

For these examples we recall that a (small) k-linear promonoidal category $(\mathscr{A}, p, j)$ (previously called "premonoidal" in [1]) consists of a k-linear category $\mathscr{A}$ and two k-linear functors

$$p : \mathscr{A}^{\mathrm{op}} \otimes \mathscr{A}^{\mathrm{op}} \otimes \mathscr{A} \longrightarrow \mathbf{Vect}$$
$$j : \mathscr{A} \longrightarrow \mathbf{Vect}$$

equipped with associativity and unit constraints satisfying axioms (as described in [1]) analogous to those used to define a monoidal structure on $\mathscr{A}$. The notion of a symmetric promonoidal category (also introduced in [1]) was extended in [3] to that of a braided promonoidal category.

The main point is that (braided) promonoidal structures on $\mathscr{A}$ correspond to cocontinuous (braided) monoidal structures on the functor category $[\mathscr{A}, \mathbf{Vect}]$. This latter monoidal structure is often called the convolution product of $\mathscr{A}$ and $\mathbf{Vect}$.

Example 5.1. Let $(\mathscr{A}, p, j)$ be a small braided promonoidal category with

$$\mathscr{A}(I, I) \cong I = k \qquad \text{and} \qquad j = \mathscr{A}(I, -),$$

and suppose that each hom-space $\mathscr{A}(a, b)$ is finite dimensional. Let $f \in [\mathscr{A}, \mathbf{Vect}_f]$ be a very weak bialgebra in the convolution $[\mathscr{A}, \mathbf{Vect}]$. Suppose also that $\mathscr{A} \subset \mathscr{C}$ where $\mathscr{C}$ is a separable braided monoidal category with

$$p(a, b, c) \cong \mathscr{C}(a \otimes b, c)$$

naturally; we suppose the induced maps

$$(\ddagger) \qquad \int^c p(a, b, c) \otimes \mathscr{C}(c, C) \longrightarrow \mathscr{C}(a \otimes b, C)$$

are isomorphisms (e.g., $\mathscr{A}$ monoidal). We also suppose that each $a \in \mathscr{A}$ has a dual $a^* \in \mathscr{A}$. Then we have maps

$$\mu : f * f \longrightarrow f \qquad \text{and} \qquad \eta : k \longrightarrow fI$$

and

$$\delta : f \longrightarrow f * f \qquad \text{and} \qquad \epsilon : fI \longrightarrow k$$

satisfying associativity and unital axioms.

Define the functor $U : \mathscr{C} \longrightarrow \mathbf{Vect}$ by

$$U(C) = \int^{a} fa \otimes \mathscr{C}(a, C);$$

then, by the Yoneda lemma, $U(a^*) \cong U(a)^*$ if $f(a^*) \cong f(a)^*$ for $a \in \mathscr{A}$. Moreover, U is monoidal and comonoidal on $\mathscr{C}$ via the maps r and i described in the diagram:

$$\begin{tikzcd}
UC \otimes UD \arrow[r, "\cong"] \arrow[dddd, shift right, "r"'] & \int^{a,b} fa \otimes fb \otimes \mathscr{C}(a, C) \otimes \mathscr{C}(b, D) \arrow[d, shift right] \\
& \int^{a,b} fa \otimes fb \otimes \mathscr{C}(a \otimes b, C \otimes D) \arrow[u, shift right, "\mathscr{C}\text{ separable}"'] \arrow[d, "(\ddagger)"] \\
& \int^{a,b} fa \otimes fb \otimes \int^{c} p(a, b, c) \otimes \mathscr{C}(c, C \otimes D) \arrow[d, shift right, "\mu"'] \\
& \int^{c} fc \otimes \mathscr{C}(c, C \otimes D), \arrow[u, shift right, "\delta"'] \arrow[dl, "="] \\
U(C \otimes D) \arrow[uuuu, shift right, "i"'] &
\end{tikzcd}$$

Thus, if f is separable, then so is U with $\dim UI = \dim fI$ since

$$UI = \int^{a} fa \otimes \mathscr{C}(a, I) \cong fI$$

by the Yoneda lemma, so that $i_0 r_0 = \dim UI \cdot 1$ if and only if $\epsilon\eta = \dim fI \cdot 1$.

Example 5.2. Suppose that $(\mathscr{A}^{\mathrm{op}}, p, j)$ is a small promonoidal category with $I \in \mathscr{A}$ such that $j \cong \mathscr{A}(-, I)$ and with each $x \in \mathscr{A}$ an "atom" in $\mathscr{C}$ (i.e., an object $x \in \mathscr{C}$ for which $\mathscr{C}(x, -)$ preserves all colimits) where $\mathscr{C}$ is a cocomplete and cocontinuous braided monoidal category containing $\mathscr{A}$ and each $x \in \mathscr{A}$ has a dual $x^* \in \mathscr{A}$. Suppose that the inclusion $\mathscr{A} \subset \mathscr{C}$ is dense over $\mathbf{Vect}$ (that is, the canonical evaluation morphism

$$\int^{a} \mathscr{C}(a, C) \cdot a \longrightarrow C$$

is an isomorphism for all $C \in \mathscr{C}$) and

$$x \otimes y \cong \int^{z} p(x, y, z) \cdot z$$

so that

$$\begin{aligned}\mathscr{C}(a, x\otimes y) &= \mathscr{C}(a, \int^z p(x,y,z)\cdot z)\\ &\cong \int^z p(x,y,z)\otimes\mathscr{C}(a,z) \quad \text{since } a\in\mathscr{A} \text{ is an atom in } \mathscr{C},\\ &\cong p(x,y,a) \qquad \text{by the Yoneda lemma applied to } z\in\mathscr{A}.\end{aligned}$$

Let $W : \mathscr{A} \longrightarrow \mathbf{Vect}$ be a strong promonoidal functor on $\mathscr{A}$. This means that we have structure isomorphisms

$$\begin{aligned}Wx\otimes Wy &\cong \int^z \mathscr{C}(z, x\otimes y)\otimes Wz\\ k &\cong WI\end{aligned}$$

satisfying suitable associativity and unital coherence axioms. Define the functor $U : \mathscr{C} \longrightarrow \mathbf{Vect}$ by

$$UC = \int^a \mathscr{C}(a, C)\otimes Wa.$$

Then

$$\begin{aligned}U(x^*) &= \int^a \mathscr{C}(a, x^*)\otimes Wa\\ &\cong W(x^*)\\ &\cong W(x)^*,\end{aligned}$$

if $W(x^*)\cong W(x)^*$ for all $x\in\mathscr{A}$, and

$$\begin{aligned}UI &= \int^a \mathscr{C}(a, I)\otimes Wa\\ &\cong WI\\ &\cong k,\end{aligned}$$

so that $i_0r_0 = 1$ and $r_0i_0 = 1$. Also there are mutually inverse composite maps r and i given by:

$$\begin{aligned}r : UC\otimes UD &\cong \int^{x,y} \mathscr{C}(x,C)\otimes\mathscr{C}(y,D)\otimes Ux\otimes Uy\\ &\cong \int^{x,y} \mathscr{C}(x,C)\otimes\mathscr{C}(y,D)\otimes Wx\otimes Wy\\ &\cong \int^{x,y} \mathscr{C}(x,C)\otimes\mathscr{C}(y,D)\otimes \int^z \mathscr{C}(z, x\otimes y)\otimes Wz\\ &\cong \int^z \mathscr{C}(z, C\otimes D)\otimes Wz\\ &\cong U(C\otimes D),\end{aligned}$$

which uses the assumptions that $\mathscr{C}$ is cocontinuous monoidal and $\mathscr{A}\subset\mathscr{C}$ is dense. Thus $ri = 1$ and $ir = 1$ so that U is a strong monoidal functor.

Example 5.3. (See [5] Proposition 3.) Let $\mathscr{C}$ be a braided compact monoidal category and let $\mathscr{A}\subset\mathscr{C}$ be a full finite discrete Cauchy generator of $\mathscr{C}$ which contains I and is closed under dualization in $\mathscr{C}$. As in the Häring-Oldenburg case [5], we suppose that each hom-space $\mathscr{C}(C, D)$ is finite dimensional with a chosen natural isomorphism $\mathscr{C}(C^*, D^*)\cong\mathscr{C}(C, D)^*$.

Then we have a separable monoidal functor

$$UC = \bigoplus_{a,b\in\mathscr{A}} \mathscr{C}(a, C\otimes b),$$

whose structure maps are given by the composites

$$\begin{aligned}
r : UC \otimes UD &\cong \bigoplus_{a,b,c,d} \mathscr{C}(c, C\otimes b)\otimes \mathscr{C}(a, D\otimes d)\\
&\underset{c=d}{\overset{\text{adjoint}}{\rightleftarrows}} \bigoplus_{a,b,c} \mathscr{C}(c, C\otimes b)\otimes \mathscr{C}(a, D\otimes c)\\
&\cong \bigoplus_{a,b} \mathscr{C}(a, D\otimes(C\otimes b))\\
&\cong \bigoplus_{a,b} \mathscr{C}(a, (D\otimes C)\otimes b)\\
&\cong \bigoplus_{a,b} \mathscr{C}(a, (C\otimes D)\otimes b)\\
&= U(C\otimes D),
\end{aligned}$$

and $r_0 : k \longrightarrow UI$ the diagonal, with i_0 its adjoint. Also

$$\begin{aligned}
U(C^*) &= \bigoplus_{a,b} \mathscr{C}(a, C^*\otimes b)\\
&\cong \bigoplus_{a,b} \mathscr{C}(a^*, C^*\otimes b^*)\\
&\cong \bigoplus_{a,b} \mathscr{C}(a, C\otimes b)^*\\
&\cong UC^*
\end{aligned}$$

for all $C \in \mathscr{C}$.

Example 5.4. Let $(\mathscr{A}, p, j)$ be a finite braided promonoidal category over $\mathbf{Set}_f$ with $I \in \mathscr{A}$ such that $j \cong \mathscr{A}(I, -)$ and with a promonoidal functor

$$d : \mathscr{A}^{\mathrm{op}} \longrightarrow \mathbf{Set}_f$$

for which each structure map

$$u : \int^z p(x,y,z)\times dz \longrightarrow dx \times dy$$

is an injection, and $u_0 : dI \longrightarrow 1$ is a surjection. Then we have corresponding maps

$$\int^z k[p(x,y,z)]\otimes k[dz] \rightleftarrows k[dx]\otimes k[dy]$$

and

$$k[dI] \rightleftarrows k[1],$$

where $k[s]$ denotes the free k-vector space on the (finite) set s, in $\mathbf{Vect}_f$. Define the functor $U : \mathscr{C} \longrightarrow \mathbf{Vect}_f$ by

$$Uf = \int^x fx \otimes k[dx]$$

for $f \in \mathscr{C} = [k_*\mathscr{A}, \mathbf{Vect}_f]$ (with the convolution braided monoidal closed structure) so that

$$\begin{aligned}
r : Uf \otimes Ug = &\left(\int^x fx \otimes k[dx]\right) \otimes \left(\int^y gx \otimes k[dy]\right) \\
\cong &\int^{x,y} fx \otimes gy \otimes (k[dx] \otimes k[dy]) \\
\rightleftarrows &\int^{x,y} fx \otimes gy \otimes \left(\int^z k[p(x,y,z)] \otimes k[dz]\right) \\
\cong &\int^z \left(\int^{x,y} fx \otimes gy \otimes k[p(x,y,z)]\right) \otimes k[dz] \\
= &\int^z (f \otimes g)(z) \otimes k[dz] \\
= &\int^z U(f \otimes g)
\end{aligned}$$

and

$$\begin{aligned}
i_0 : UI = &\int^x k[\mathscr{A}(I,x)] \otimes k[dx] \\
\cong &\; k[dI] \\
\rightleftarrows &\; k[1] \cong k.
\end{aligned}$$

Hence $i_0 r_0 = \dim UI \cdot 1 = |dI| \cdot 1$. Thus, U becomes a separable monoidal functor.

Example 5.5. Let $\mathscr{A}$ be a finite (discrete) set and give the cartesian product $\mathscr{A} \times \mathscr{A}$ the $\mathbf{Set}_f$-promonoidal structure corresponding to bimodule composition (i.e., to matrix multiplication). If

$$d : \mathscr{A} \times \mathscr{A} \longrightarrow \mathbf{Set}_f$$

is a promonoidal functor, then its associated structure maps

$$\begin{aligned}
\sum_{z,z'} p((x,x'),(y,y'),(z,z')) \times d(z,z') = &\sum_{z,z'} \mathscr{A}(z,x) \times \mathscr{A}(x',y) \times \mathscr{A}(y',z') \times d(z,z') \\
\cong &\; \mathscr{A}(x',y) \times d(x,y') \\
\longrightarrow &\; d(x,x') \times d(y,y'),
\end{aligned}$$

and

$$\begin{aligned}
\sum_{z,z'} j(z,z') \times d(z,z') = &\sum_{z,z'} \mathscr{A}(z,z') \times d(z,z') \\
\cong &\sum_z d(z,z) \\
\longrightarrow &\; 1,
\end{aligned}$$

are determined by components

$$\begin{aligned}
d(x,y') &\rightarrowtail d(x,y) \times d(y,y') \\
d(z,z) &\twoheadrightarrow 1
\end{aligned}$$

which give $\mathscr{A}$ the structure of a discrete cocategory over $\mathbf{Set}_f$.

Define the functor $U : \mathscr{C} = [k_*(\mathscr{A} \times \mathscr{A}), \mathbf{Vect}_f] \longrightarrow \mathbf{Vect}_f$ by

$$Uf = \bigoplus_{x,y} (f(x,y) \otimes k[d(x,y)]).$$

Then we obtain monoidal and comonoidal structure maps

$$U(f \otimes g) \underset{i}{\overset{r}{\rightleftarrows}} Uf \otimes Ug$$

$$UI \underset{i_0}{\overset{r_0}{\rightleftarrows}} k \cong k[1]$$

from the canonical maps

$$\bigoplus_{x,y,z} f(x,z) \otimes g(z,y) \otimes k[d(x,y)]$$

$$\underset{z=u=v}{\overset{\text{adjoint}}{\rightleftarrows}} \bigoplus_{x,u} \big(f(x,u) \otimes k[d(x,u)]\big) \otimes \bigoplus_{v,y} \big(g(v,y) \otimes k[d(v,y)]\big)$$

and

$$\bigoplus_{z} k[d(z,z)] \rightleftarrows k \cong k[1].$$

These give U the structure of a separable monoidal functor on $\mathscr{C}$.

6. Concluding remarks

If the original "fibre" functor U is faithful and exact then the Tannaka equivalence (duality)

$$\mathrm{Lex}(\mathscr{C}^{\mathrm{op}}, \mathbf{Vect}) \simeq \mathbf{Comod}(\mathrm{End}^\vee U)$$

is available. Thus, since $\mathscr{C}$ is braided monoidal, so is $\mathbf{Comod}(\mathrm{End}^\vee U)$ with the tensor product and unit induced by the convolution product on $\mathrm{Lex}(\mathscr{C}^{\mathrm{op}}, \mathbf{Vect})$; for convenience we recall [2] that, for $\mathscr{C}$ compact, this convolution product is given by the restriction to $\mathrm{Lex}(\mathscr{C}^{\mathrm{op}}, \mathbf{Vect})$ of the coend

$$\begin{aligned} F * G &= \int^{C,D} FC \otimes GD \otimes \mathscr{C}(-, C \otimes D) \\ &\cong \int^{C} FC \otimes G(C^* \otimes -) \end{aligned}$$

computed in the whole functor category $[\mathscr{C}^{\mathrm{op}}, \mathbf{Vect}]$. Moreover, when U is separable monoidal, the category $\mathbf{Co}(\mathrm{End}^\vee U)$ of cofree coactions of $\mathrm{End}^\vee U$ (as constructed in [6] for example) also has a monoidal structure $(\mathbf{Co}(\mathrm{End}^\vee U), \otimes, k)$, this time obtained from the algebra structure of $\mathrm{End}^\vee U$. The forgetful inclusion

$$\mathbf{Comod}(\mathrm{End}^\vee U) \subset \mathbf{Co}(\mathrm{End}^\vee U)$$

preserves colimits while $\mathbf{Comod}(\mathrm{End}^\vee U)$ has a small generator, namely $\{UC \mid C \in \mathscr{C}\}$, and thus, from the special adjoint functor theorem, this inclusion has a right adjoint. The value of the adjunction's counit at the functor $F \otimes G$ in $\mathbf{Co}(\mathrm{End}^\vee U)$ is then a split monomorphism and, in particular, the monoidal forgetful functor

$$\mathbf{Comod}(\mathrm{End}^\vee U) \longrightarrow \mathbf{Vect},$$

which is the composite $\mathbf{Comod}(\mathrm{End}^\vee U) \subset \mathbf{Co}(\mathrm{End}^\vee U) \longrightarrow \mathbf{Vect}$, is a separable monoidal functor extension of the given functor $U : \mathscr{C} \longrightarrow \mathbf{Vect}$.

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